THE

MATHEMATICAL GAZETTE.

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc.; PROF. H. W. LLOYD-TANNER, M.A., D.Sc., F.R.S.;
PROF. E. T. WHITTAKER, M.A., F.R.S.

LONDON:
G. BELL AND SONS, LTD., PORTUGAL STREET, KINGSWAY,
AND BOMBAY.

VOL. VI. JANUARY, 1912. No. 96.

The Mathematical Association.

THE FUTURE CONSTITUTION OF THE TEACHING COMMITTEES OF THE ASSOCIATION.

In view of the growth and development of the Association in recent years, and the wider interests which it now represents, it is desirable that the Teaching Committee should be re-constituted on a broader basis.

With this object the following regulations have been made for the appointment of a General Teaching Committee and three Special Committees:

I. Formation of Special Committees.

Special Committees for the consideration of problems peculiar to different sections shall be constituted as follows:

1. A Committee of twelve Members of the Association, who are masters teaching in Public Schools, shall be elected as follows:

(*a*) Members and Associates of the Association shall be invited to propose members for election. The number proposed by any one individual may not exceed twelve.

(*b*) A voting paper containing the names of all those thus proposed shall be sent to all Members and Associates of the Association who are masters at Public Schools, each of whom may vote for not more than twelve of those proposed.

(*c*) The twelve obtaining the highest number of votes, and who are willing to serve, shall constitute the Public Schools Committee. In the case of a tie, the Council shall decide.

(*d*) The Committee shall have power to co-opt not more than three others, who need not necessarily be Members or Associates of the Association at the time of their election, but who shall have no vote until they have joined the Association either as Members or as Associates.

NOTE.—By a "Public School" is meant a school which has a place in the Public Schools Year Book.

2. A similar Special Committee shall be constituted in like manner for "Other Secondary Schools for Boys," *i.e.* for Secondary Schools which have no place in the Public Schools Year Book.

3. A similar Special Committee shall be constituted in like manner for "Girls' Schools," except that all lady Members of the Association shall be eligible for membership of the Committee, and lady Members and Associates may propose and vote whether engaged in school teaching or not.

NOTE.—Any two, or all, of the Special Committees may sit together for the consideration of any matter of common interest.

II. Constitution of the General Committee.

The General Committee shall consist of:

(*a*) 6 representatives of Public Schools.
(*b*) 4 " " Other Secondary Schools for Boys.
(*c*) 6 " " Girls' Schools.
(*d*) 2 " " Technical Schools.
(*e*) 2 " " Preparatory Schools.
(*f*) 2 " " those engaged in training teachers.
(*g*) 3 " " Universities.
(*h*) 5, or fewer, other Members nominated by the Council.
(*i*) 5, or fewer, other persons co-opted by the Committee.
(*j*) The President and Honorary Secretaries of the Association.

All members of the General Committee, except the co-opted members, shall be Members of the Association.

1. Representatives under (*a*) shall be the 6 Members of the Special "Public Schools" Committee who received most votes in the election of that Committee.

2. Representatives under (*b*) shall be the 4 Members of the Special "Other Secondary Schools for Boys" Committee who received most votes in the election of that Committee.

3. Representatives under (*c*) shall be the 6 Members of the Special "Girls' Schools" Committee who received most votes in the election of that Committee.

NOTE.—In the case of a tie, or of an uncontested election, the Council shall decide.

4. Representatives under (*d*), (*e*), (*f*), and (*g*) shall be nominated by the Council.

III. General.

1. The first election shall take place as soon as possible after the annual meeting in 1912, and the committee so elected shall hold office till December, 1913.

2. The methods of election for the general committee and special committees, as described above, shall be for the first election only; the methods of future elections shall be re-considered by the Council before July, 1913.

3. The Council shall appoint conveners for the first meetings of the General and Special Committees as constituted above; at these meetings the committees shall elect their own officers.

IV. Reports.

1. The General Committee may draw up its own reports; it may appoint sub-committees, and may submit questions to one or more of the Special Committees. Reports of the General Committee shall be recommended to the Council for publication. Unanimous reports, if published, shall be published as Reports of the Committee. Other reports shall be signed by those members of the Committee who approve of their contents, and, if published, shall be published as the reports of the signatories only.

2. A Special Committee may draw up reports on questions relating to matters within its province. Such reports shall be signed by those members of the Special Committee who approve of their contents, and shall be then forwarded to the General Committee. The General Committee may (*a*) recommend such reports to the Council for publication as reports of one or more Special Committees, or (*b*) adopt them, with the consent of the Special Committee or Committees, and recommend them to the Council for publication as reports of the General Committee, or (*c*) refer them back to the Special Committee or Committees.

3. Reports recommended to the Council for publication (as above) shall be published by the Council unless the Council considers that it is inexpedient to do so for financial reasons.

EXHIBITION OF MODELS AT THE INTERNATIONAL CONGRESS.

To the Editor of the *Mathematical Gazette*.

INTERNATIONAL CONGRESS OF MATHEMATICIANS.

January 20th, 1912.

Dear Sir,—I regret to say that there has been very little response to the notice which was inserted in the last number of the *Gazette* respecting the exhibition of models, apparatus, books, etc., which the Association is organising in connection with the International Congress. The committee charged with the organisation of the exhibition has gone further into the matter, and I am able to give a few details concerning the various sections in which it is proposed to arrange the exhibition. These may be classified as follows:

1. *Models and Diagrams.* In this section it is proposed to make a collection of models, made by the pupil or teacher, for use or illustration in the teaching of mathematics or mechanics. The committee desire that as far as possible these should not be mere isolated models, but should illustrate some connected piece of work. An explanatory paper should accompany the models, and it is proposed that these shall be printed for distribution to visitors. Offers of exhibits in this section should be sent as soon as possible to P. Abbott, 5 West View, Highgate Hill, N.

2. *Exhibition of Work.* The committee think that it would prove of great interest if it could be possible to show the actual note books, or work books of the pupils where some special scheme of work is being carried out or some experiment is being made in mathematical teaching. It is not intended that mere show books should be exhibited, but genuine specimens of class work. Offers from public schools and secondary schools should be sent to F. Boon, Dulwich College.

3. *Exhibition of Mathematical Text-books.*

4. *Exhibition of Mathematical and Mechanical Apparatus* (from makers). The organisation of these is being undertaken by Mr. Abbott.

5. *Specimen Syllabus.* It was felt by the committee that visitors to the congress would be interested in syllabuses of mathematical teaching in those schools where attempts are being made to develop the teaching on modern lines. Offers of help will be welcomed.

There are two or three other directions in which the committee hope to make arrangements for exhibits, but I trust that what I have written may serve to remind members of the necessity for taking the matter in hand at once. Suggestions will be welcomed by the committee. I am,

Yours faithfully,

P. Abbott,
Hon. Sec. of the Committee.

5 West View,
Highgate Hill, N.

GEOGRAPHICAL DISTRIBUTION OF MEMBERS.

MR. H. D. ELLIS has been good enough to compile the following statistics: A list of members of the Association on January 1st, 1911, classified according to locality, shows a total of 680, of whom England contains 531, Wales 21, Scotland 23, Ireland 14, Isle of Man 1, Channel Islands 2, South Africa 23, Canada 6, Australia 9, New Zealand 4, West Indies 1, Malta 1, India 17, Brazil 1, Egypt 3, Italy 1, Japan 2, Roumania 1, Syria 1, U.S.A. 18.

The 531 members in England are distributed thus: London 146, Bedfordshire 7, Berkshire 13, Bucks. 12, Cambridge 31, Cheshire 6, Cornwall 2, Cumberland 5, Derbyshire 4, Devonshire 11, Dorset 1, Durham 4, Essex 11, Glos. 26, Hants. 19, Herts. 12, Hereford 1, Isle of Wight 2, Kent 23, Lancashire 25, Leicester 6, Lincs. 1, Middlesex 8, Monmouth 1, Norfolk 8, Northants. 3, Northumberland 4, Notts. 3, Oxford 13, Rutland 4, Shropshire 6, Somerset 8, Staffs. 4, Surrey 21, Sussex 18, Warwickshire 13, Wilts. 2, Worcs. 9, Yorkshire 38.

LOCAL BRANCHES.

NORTH WALES BRANCH.

A MEETING of this branch was held on November 18th at the Bangor County School for Girls, where Miss Mason, the headmistress, entertained the members present.

Prof. Bryan opened a discussion on Continuity in Teaching between School and University, urging that the practical side of education should be kept in the forefront, and that there should be more interchange of views between teachers and examiners. He mentioned some objections to the present system of appointing fresh external examiners of a school every two or three years, just when the old ones were getting into touch with the school's work, recommending rather that schools should be examined by university professors themselves, thus bringing them into closer contact with school methods of work, or that at any rate school examination papers should be revised by university professors.

Miss Mason was strongly in favour of Boards of Studies for both elementary and secondary schools, and thought that informal conferences between school teachers and university professors would be of great advantage. Universities ought to lead, and to suggest what should be done in schools.

Mr. Madoc Jones (Beaumaris) said that, in the United States, university professors visited schools before examining them, and advocated such meetings as those mentioned above, on the ground that teachers might be kept in touch with modern tendencies and avoid falling into grooves.

Some discussion followed on the amount of Pure Mathematics desirable in examination papers, and it was generally agreed that more questions in Geometrical Conics should be set on account of the application of the subject afterwards.

SYDNEY BRANCH.

THE branch has 40 members, with Professor Carslaw as President. Meetings are held half-yearly in one of the class rooms of Sydney University. The first regular meeting was held in December 1910, when the President read a paper on the Teaching of Mathematics in Secondary Schools.

The State Government has recently received a syllabus of work for Secondary Schools. At the meeting in May of the local branch, a discussion took place on a memorandum on the mathematical work of this syllabus, drawn up by Professor Carslaw, special reference being made to the teaching

of Geometry. The discussion was opened by Mr. J. D. St. Clair Machardy and Professor Newham. Arrangements were made for distributing the *Mathematical Gazette* among the members.

At a meeting of the Sydney Branch, held in Sydney on November 17th, the following officers were elected for 1912. President, Professor H. S. Carslaw, D.Sc., University of Sydney; Treasurer, Mr. G. F. Wooldridge, M.A., The King's School, Parramatta; Secretary, Mr. R. J. Middleton, M.A., Teachers' College, Sydney. The President delivered an address on "Some Branches of Modern Mathematics of Special Interest to the Mathematical Teachers in the Schools." The Branch consists of 5 Members of the Association and 37 Associates, and holds two meetings yearly.

R. J. MIDDLETON,
Acting Hon. Secretary, Teachers' College, Sydney.

THE PLACE OF MATHEMATICS IN GIRLS' EDUCATION.

AT a meeting of the London Branch of the Association, on December 9, a paper on the above question was read by Miss S. A. Burstall, Head Mistress of the Manchester High School for Girls. Mr. C. S. Jackson presided, and there was a large attendance of members. Miss Burstall's paper was as follows:

Ladies and Gentlemen: While I thank you very sincerely for the honour you have done me in the invitation to speak before you this afternoon, I nevertheless tremble at the task which lies before me; for the part I have to play is that of the Adversary, who questions the value in education of the very subject you are combined to support. I do indeed need courage to express to you my profound conviction, based though it be on experience, that the place of Mathematics in girls' education should, except in the case of a small but distinguished minority, be a wholly subordinate and limited one. To put the matter in another form, I definitely challenge the position of compulsory Mathematics in Matriculation and College Entrance Examinations, as some do that of compulsory Greek at the older universities, or as many would attack compulsory Latin in Matriculation. Personally, indeed, I should prefer compulsory Latin to compulsory Mathematics for girls. It appears to me that an error has been made in the development of girls' education, by introducing Mathematics so completely and fully into the regular curriculum for the average girl. I make no assertion about the education of boys. The position may be wholly different for them. Indeed I am inclined to think that there is a much greater natural difference between boys and girls of secondary school age than is at present recognised, a difference largely, of course, physical, but so fundamental that it affects intellectual work and every department of school life. The modern system of the education of girls has been attacked on the ground that it has been modelled too

closely on that of boys: that the pioneers in women's education, instead of striking out a new line for themselves and establishing a system of higher education suited to the needs of girls and women, simply took over the existing system as developed for boys and men. The facts are true, but the reproach is not quite just. At the beginning of the whole movement, which I am old enough to remember, it was necessary for the reformers to take the educational system as they found it, and to demand for girls an entry into the general intellectual inheritance from which so far they had been excluded. Only in this way could prejudice be removed, only thus could women justify the demands for higher education. If we adopt Nicholas Murray Butler's definition that education is the process of entering into the spiritual inheritance of the race, we must acknowledge that, on the whole and broadly speaking, there must be a common inheritance of culture and civilisation for both men and women. It is also important that the intellectual interests of the two sexes should not be so hopelessly different and separate as they were before the days of the reform. It is of the highest importance indeed, socially, that men and women should be able to understand one another and to meet on a common intellectual ground.

However, this does not necessarily mean that all boys and all girls must study exactly the same subjects to the same degree of proficiency. Many of us, while we gratefully recognise the value of the work of the reformers, and do not wish in any sense to undo it, are convinced that the time has come for a change, for a further development, for a differentiation between the school curriculum of boys and girls. We venture to question the accepted doctrine that a certain amount of mathematical study is an essential part of a liberal education. We know, of course, that we are heretics, but we are consoled by the recollection that the heterodoxy of to-day is often the orthodoxy of to-morrow, and that the pioneer women in the 'sixties were far greater heretics, for the time being, than we.

Before I enter on my main statement, you will forgive, perhaps, a personal reference. I was, as a girl, a humble student of Mathematics. I rejoiced in it as a new world of joy and beauty, a world opened to me, more than thirty-five years ago, by the brilliant mathematical teacher who now presides over the destinies of the Frances Mary Buss School. I took the Mathematical Tripos as long ago as 1881, and more or less I have taught the subject ever since to every type of pupil, Scholarship girls, Matriculation classes, and weak sets in the Middle School. Perhaps it is only one who herself has taught and studied Mathematics that would dare to express what I am endeavouring to put before you.

The first point I would make is that my conviction has arisen

from experiences, more especially experience as a headmistress during the past thirteen years in a school somewhat fortunately situated, in that it draws from an area where there is considerable intellectual vigour and interest. Were my experience that of working in rural areas, or with a less hard-headed type of population than that of the North, it would count for less, in that the material taught might be considered less apt for abstract studies. I have found that an even moderate degree of success in mathematical study among girls, the average girls that is, can only be obtained at an excessive cost in time, energy, and teaching power, a lesson and home work every day of the week for at least four or five years, or even six years. This involves often the exclusion of subjects which would be far more profitable and useful to the girls. Occasionally there is so much strain and effort about the work that, in consideration of her future health, the girl is obliged to give up the idea of going to college, when in every other respect but mathematical capacity she is eminently suited for a college career. It is generally agreed that girls work harder than boys, and that in good Girls' Schools the teaching is not inferior to that given by men; but it is also generally recognised by those in a position to know that the mathematical results are very much worse than with boys. I would appeal to the statistics of public external examinations (the proportion of girls who fail in Mathematics is always larger), and to the experience of teachers in mixed schools. I can quote two witnesses by name—Mr. Iliffe, of Sheffield, and Mr. Crosthwaite, the headmaster of our Manchester Municipal Secondary School, who conducted some years ago a careful inquiry into the relative successes of boys and girls in his school in different subjects. The inferiority of the girls' Mathematics was marked. I would also appeal to the experience of the homes where boys and girls work side by side. Parents know well how much more effort and trouble their girls have over mathematical study. Speaking as an organiser, I grudge all this effort, for which so scanty a return is secured. We ought to recognise that the average girl has a natural disability for Mathematics. One cause may be that she has less vital energy to spare during the years of Secondary School life. She has to grow, and she should start adult life with a sound reserve of physical strength. Furthermore—a fact not generally recognised—there is a greater drain on her energy while at school. She is expected to do much more at home than is expected from a boy, and when any emergency, sickness or the like, arises, it is the girl who has to stand in the breach and help at home and nurse her relatives. This, hard as it is on her, is quite right, and we ought to recognise this natural burden, and not require from her also proficiency in the most exhausting and difficult of school studies. We ought also to recognise that most girls (I am speaking now

strictly of the average, not of the one in a thousand who will be a mathematical genius) should give more time to social duties and to the study of art and music than should boys.

All this leads me to my second point—the problem of the crowded curriculum—with which we all struggle, but which, as I have suggested, is far worse for girls than for boys. A girl in the Fifth or Sixth Form has to take the same examinations as her brother, and compete, not for the same University and College Scholarships, but for a far scantier supply. In addition she should have, and in most cases does have, home duties, and she is expected to give some time to accomplishments. At the same time her health needs a much greater care and consideration than that of a boy at the same age; and this, not in her own interests merely, but in the interest of the race, of the future. If our girls are to be ready for all their responsibilities, public and private, in the days to come, their work at school *must* be lightened: *something must go.* It is my profound conviction that the sacrifice should be Mathematics; it is not really needed for girls. The parents are quite right when they say: "I do not see why my girl should learn Algebra, it will be of no use to her." The question for boys is wholly different. Mathematics, as you all know, is closely related to the interests and characteristic industries of men, more particularly to those employments which women naturally would not share: war, seamanship, exploration, engineering of all kinds, surveying and mining, building and workshop practice, and finance. Every boy, whatever his future business, will have some relation to these characteristic industries and occupations, and some sound knowledge of Mathematics is in his case necessary. On the other hand, we have found by experience that Biology and Psychology make, in the intellectual sphere, a very strong appeal to girls and women. They study these sciences with ease and delight, and have already achieved remarkable success. These sciences are closely related to women's characteristic activities, in the nurture of the young of the race, and in the management of personalities in the home. They thus come to these subjects with a natural predisposition, as they certainly do not in the case of Mathematics.

I do not forget that there are certain elementary mathematical concepts and manipulations which are a necessary part of our intellectual inheritance, and are actually required for the conduct of life—arithmetic of a practical kind and some geometrical training, possibly even a little algebra as an ancillary subject in science study. But the ordinary course of study in many girls' schools and the demands of Matriculation examinations go far beyond this minimum, and involve, I say it again, what seems to me a terrible waste of power, and a diversion of the average girl's energy from fields where it might be spent with far greater profit.

Some will answer my attack by saying that the value of Mathematics is in the training that it gives, more particularly the training in reasoning power and accuracy. There is, of course, some truth in this, but psychologists tell us that we can no longer appeal to the old-fashioned faculty psychology and suppose that particular subjects train different parts of the mind, or that there are "separate faculties or powers in the mind or the muscles ready to be trained." The mind is trained as a whole. The exceptional girl who has a gift for Mathematics, who can grasp the material which it provides, undoubtedly does receive valuable mental training from the study. Her powers are exercised and developed by it, but this is because she has taste, and not because of the peculiar merits of the subject. Beans and cheese are nourishing and cheap foods if you can digest them. If you cannot, they are far better left on your plate, and no expert in dieting would force them on persons whose digestion was not robust. People get mental training from subjects that they can grasp and assimilate, that have reference to their real interests, that they can work in freely. That subject is truly educative that the pupil can grasp, that inspires in her a real interest; it is only that truth which we can understand and follow that can win our devotion and develop our powers.

My third point is an argument from history. After the Schools Inquiry Commission in 1867, mathematics was arbitrarily introduced into girls' education as a definite remedy for a definite disease. The introduction of Mathematics was not a natural growth and development from the traditional education and activities of women. It had no relation indeed to their life at all. I quote from a pamphlet of one of the reformers. "This study (Mathematics) offers peculiar advantages for the correction of the mental errors to which the neglect of real culture has made women liable." The Commission itself thus described the errors of the schools as they were in the sixties. "Want of thoroughness and foundation; slovenliness and showy superficiality; inattention to rudiments; undue time given to accomplishments, and these not taught intelligently or in any scientific manner; want of organisation." The study of Mathematics was recommended by the experts as the panacea for these weaknesses. It was said, too, that women could not reason; they were to be taught how by studying Euclid. The sweetly feminine feeble creature of the early Victorian novels, and of real life, was to be braced and strengthened, and made a useful sensible human being, by grappling with equations, sines and cosines. All this was based on the psychology then current. We may compare with this artificial panacea the development of the teaching of literature and history in girls' schools. These were traditional subjects of instruction: they were carried on into the new time, strengthened

and liberalised by the influence of a broader culture given to women at college. *They* have become subjects for the exercise of reasoning powers, judgment, thoroughness. They were natural and in a sense easy; hence has arisen the ease and excellence of the English and history work in girls' schools to-day, and (may I say it) the superiority of such work to the average attainment of that in boys' schools. We may compare what has happened in the case of another subject, also introduced into girls' education with the idea of training the reason. Many of the reformers, as the original records show, introduced Latin more diffidently than was the case with Mathematics, and the subject has not become compulsory in all girls' secondary schools. Now what has happened in this case? A result wholly unforeseen; the study of classics has been followed by girls and women with remarkable success. They have seized with avidity on Greek and Latin as culture, as the humanities which make a strong appeal to women's literary interests. In some of the best schools classics have taken for the ablest girls as good a place as in the boys' schools. One may go further; original work is done by women even in the oft-reaped fields of classical study. Now, this success of Latin has been made possible by freedom. The subject has not been forced on the schools, and those girls who could follow it with success have been able to do so. The work of the reformers of the seventies *has* been justified by the results of the last thirty years, but we must adapt it and modify it for the needs of to-day. Mathematics, like Latin, has been of the highest value in the case of an exceptional minority of girls and women who have taste for the subject, and who profit by its study.

I would now pause for a moment to consider the case of this minority, and I would remind you that the mathematical students and teachers belonging to this society are *ex hypothesi* members of such a group. They are not the average, and what I would ask them in any general study of education is to remember this fact carefully. We should consider girls of every type, not merely the few with special powers. It is, of course, those few who become mathematical leaders, just as it is the brilliant classical men who become headmasters of boys' public schools. This circumstance has for long blocked the way of educational reform for boys in England. The brilliant classical men who gained much from the study of classics are "in the seats of the mighty," and not unnaturally use their power to continue the system which was so satisfactory in their own case.

In dealing with the question before us this afternoon, we must remember there is this exceptional minority of mathematical girls and women, whom I would divide into two groups. Of the first, Mrs. Somerville is a type; the woman who has a natural gift, amounting perhaps to genius, for original mathematical study.

Such persons should, of course, receive the cultivation which their talents require, if only for the sake of the additions to knowledge which they will make if they are properly educated. There are already well-known examples. The distinguished woman astronomer after whom the Vassar Observatory is named is one. Such persons are, of course, very few, and it might be said that the schools need not trouble about them. They will emerge in any case. But the other group, though not people of genius, are also important, and for those the schools may well be expected to provide. I would call them the organisers. A very fair proportion of able women have distinct gifts as organisers and administrators. The virtuous woman of the Book of Proverbs is a case in point, and there are many about us in the world to-day. They often as girls show a marked talent for mathematics; the subject appeals to their instinct for form, order, and arrangement. They profit to a remarkable degree by the course of education which ends in a Mathematical Tripos. They become more useful citizens afterwards, because of their education, which has made everything else that they do easy in comparison. I would go so far as to say that in the event of a great strain being thrown on our nation by a European war, such women might do useful service at headquarters by organising transport and food supply and leaving the men free for work at the front. Mathematical training has done much for this type of women, and the opportunities of college study so hardly won must be preserved for their sakes, as well as for the sake of the genius, the discoverer. In girls' schools, therefore, we must have provision for teaching Mathematics to quite a good standard for the sake of those few whom the schools must put on the right road. In their case Mathematics has a quite definite relation to the activities of later life; that organising and administrative work which will, I think, increasingly be opened to women who have the capacity for it, both in public and private spheres of action. Personally, I look forward to a time when mathematical study will become what Latin and Greek are in some girls' schools, the discipline of a select group who attain real excellence, worth the effort that has been spent upon it.

In conclusion, I would summarise the practical reforms that I would advocate:

I. No compulsory Mathematics in Matriculation examinations.

II. An elementary course of reformed mathematical teaching for girls in the middle school, the emphasis being on geometry rather than algebra, and the whole subject being taken in very close relation to its practical aspects.

III. Exemption at 15-16 for the average girl from any mathematical study. She should then give attention to special subjects.

IV. Continuation of the existing system for the exceptional minority who will go on to College and take honours in Mathematics.

The time and energy which would be available for school use by the reduction of the amount of Mathematics taught, and by the removal of compulsory examinations, would enable headmistresses to solve what is at the moment almost an insoluble problem, how to prepare their girls for life, and at the same time enable them to pass the examinations on which the entry to professional work depends. I would refer to two articles by the president of the Headmistresses' Association, Miss Douglas, sent forth with all the authority of an official publication in the book recently issued by our body, *Public Schools for Girls*. Some persons, however, whose opinion is worthy of respect and consideration, consider that a Matriculation examination should contain something of the nature of a stiff hedge or a big water-jump to check the duffers. Mathematics does thus act. To put the matter in a more academic fashion, a college entrance examination should contain an austere and arduous intellectual element. There is much truth in this, but the requirement can be met in more than one way. Latin is enough for a test. In the case of girls I should like to see Harmony take a place as an alternative for algebra. Most girls have to learn music, and proper harmony is just as hard as mathematics, and much more practical and direct in relation to what a girl has to do.

But all the school examinations are becoming too hard; they are throttling English education. We must get free; we must preserve the variety and vitality that come of freedom. We touch here on a big question; English education wants reforming badly. It is too formal, too completely detached from life. It is not really liberal; it certainly is not vocational. Your Association has worked for years to reform the teaching of Mathematics, to make it a living, practical reality, not a mass of academic exercises. You have done much for the boys, to whom sound mathematical training is a necessity. Could your schemes be adapted freely for girls' education, apart from the bondage of examination, much that is now a burden and a task might become a joy.

Discussion.

Miss Punnett in opening the discussion said that, although the beginning of Miss Burstall's address made her expect to be entirely antagonistic to her views, she found herself at the end of it in a position of considerable agreement with them. She could in fact accept Miss Burstall's conclusions almost without reservation, provided that they might be taken as referring to the present state of affairs only. It could not be

denied that as things are at present, many girls are finding mathematics uncongenial to them, and are consequently, to a large extent, wasting their time in studying the subject. But was it certain that this condition of affairs was due to inherent incapacity for mathematics on the part of these girls? Might it not be owing rather to the comparative failure up till now to present mathematics to girls in such a way as to attract and interest them? Much had been done in the case of boys to base their mathematics on the practical activities (for example, engineering and surveying) which are naturally interesting to boys; the same office must be performed for girls. In other words, mathematics for girls, instead of following on the same lines of interest as for boys, must be closely related to the practical activities proper to girls and naturally interesting to them. If this could be completely and effectively done, the results of teaching mathematics to girls on these lines might in a few years be such as would surprise Miss Burstall and cause her entirely to alter her view. Under such conditions, "the average girl" might possibly show herself as much interested in mathematics as "the average boy," and might work at it with as much profit as he—the speaker for her part firmly believed that this would be so. At any rate she pleaded for a suspension of judgment in the matter, and urged that those who wished to give mathematics a quite subordinate position in the education of girls should at least wait until the experiment had been fairly tried of teaching it on lines really adjusted to the interests and activities of girls.

In conclusion, Miss Punnett reminded the audience that Miss Burstall had commented on the beneficial effect of mathematics on women who possessed marked power of organisation and administration. Was it not reasonable to expect that, since administration and organisation were in certain directions—particularly in the ordering of the home—women's special province, the value of mathematics would, if the subject were properly taught, make itself felt in the lives not only of the exceptions but of the great majority of women?

Miss C. Waters said her experience differed from Miss Burstall's on certain points (*a*) Three years in a mixed school showed little or no difference between girls' and boys' mathematical power, though perhaps the girls never reached quite the depths of stupidity attained by one or two boys. This might be accounted for by disinclination on the part of the boy rather than lack of ability. (*b*) The physiological strain of adolescence is as great for boys as for girls, and the danger of overwork quite as grave. At the school referred to above, the breakdowns that did occur were among the boys, but overstrain was apparent in both. Girls' home duties are urged in support of the necessity for

lightening the school work, but whereas Miss Burstall would recognise these as essential she (the speaker) would not, unless they could be equally divided between boys and girls as in America. Latin is not common in the newer secondary girls' schools, and consequently if mathematics be removed there does not remain a single subject that demands "grit."

In conclusion, she gravely feared that any differentiation between boys' and girls' education would produce a serious set-back in the latter. It would come to be regarded as inferior, merely because it is woman's not man's, and we should lose much of the advantage won for us by the pioneers.

Miss M. Cross (King's Langley Priory) said: The difficulty which is being felt as to the place of "Mathematics in Girls' Schools" is due not so much to the nature of the subject as to the method in which it has been treated.

Owing partly to the wholesale adoption in girls' schools of the curriculum as it has existed for boys, partly to the general reference in school to the requirements of public examination, and partly to the failure on the part of many teachers to recognise the true meaning of mathematical study, instruction has proceeded along lines bound sooner or later to result in serious injury to faculty.

Thus, geometry has been conceived of mainly as a process of reasoning, and children have too often been asked to demonstrate principles, the underlying relations of which have never come into their experience.

Similarly, algebra has been regarded as the science in which number as *quantity* has to be investigated.

Now mathematics is the one branch of human knowledge which, more than any other, is independent of the (so-called) external world; it follows then that its laws are the laws of the working of the human mind, and that mathematical truth must be derived from *within*, not imposed from *without*.

When teachers clearly understand that in this subject, to a degree greater than in any other, their office is to direct (to some extent) the course of enquiry, to provide short cuts when desirable, and, for the rest, to stand aside and allow the class to evolve its own formulae, its own demonstrations, its own syntheses; then, I feel sure, it will be found that girls, with their quick perceptions, will make at least as ready a response as boys to the vitalising influence which works through the agency of real mathematics.

There can be no strain in work of this kind. Indeed, experience rather goes to show that, wherever fatigue or exhaustion has been set up, this has been Nature's revolt against the system by which the mere body of mathematical truth has been divested of its spiritual significance, a system which has too long found a place in our class rooms.

Mrs. Jessie White said that she agreed with the last speaker but one, that there was no essential difference between boys and girls in respect to mathematical power. She had taught both. She took exception to Miss Waters' remark that if Latin and mathematics were removed from the curriculum there would be no subject left capable of developing grit. Miss Waters had forgotten science, and in her opinion no subjects were more calculated to call out grit than physical and chemical science. She had in her hand the Interim Memorandum of the Teaching of Housecraft in Secondary Schools, a very interesting document because of the information it gave about the time-tables of a good many schools. In only one school was the total time throughout the school course given to science greater than that given to mathematics, in most schools it was very much less. When it was remembered how large a part of the science time was devoted to the biological sciences, the disproportion became much greater. Girls could be interested in physics, and physics could be brought into closer connection than had been done with the interest in the house which every girl felt or ought to feel. Such teaching could be begun at eleven and would supply the motive for mathematical study, a motive which was absent from the mathematical study of the majority of girls to-day. In such a course those who had mathematical ability could be distinguished from those who had little. There were girls who could not possibly do a geometrical rider. Miss Burstall had made the illuminating remark that it is what we can do that educates us, not what we cannot do. In this connection Mrs. White referred to a pedagogical article which she had just been reading which dwelt on the importance of self-confidence in securing success in life. Girls who, throughout a long course, had been set to do what they could not do, were not likely to have their self-confidence developed. The advantage of giving more time to physics than to mathematics in the earlier years was that physics made an appeal to more varied powers and interests than mathematics did.

Lastly, the study of geography involved also a certain amount of mathematical study, and formed a good introduction to the latter study.

Miss A. Somers said that in her experience both girls and boys, especially the former, who left elementary schools at 14, without previous mathematical teaching beyond arithmetic, frequently showed a great interest and keenness in attacking a new and therefore especially interesting subject such as mathematics. The interest thus shown was much in advance of that evinced by children of the same age who had already studied geometry and algebra for two years or even more.

THE AIM AND METHODS OF SCHOOL ALGEBRA.[1]

II. METHODS OF TEACHING.

§ 1. *The rhythm of mathematical progress.* To the question how the aim described in the preceding lecture is to be realised, the whole of the rest of the course is proffered as an answer. Here it will be sufficient to summarise the main features of the methods which are later to be given in detail.

The professed aim is to make school mathematics a reproduction, as faithful as the difference of the situations permits, of the mathematical activities of the great world. The method must, then, be based upon an analysis of the general course of development which these activities display. Remembering, as before, that this kind of analysis always introduces an artificial simplicity into a naturally complicated matter, we may yet lay it down that mathematics advances by the constant repetition of a normal cycle of progress—a cycle in which three typical phases may be distinguished. In the first, the *heuristic* phase, the mathematician, face to face with a new type of problem, devises a new notation or mathematical method to deal with it. In the second, the *formal* phase, the new notation or method is studied apart from the immediate needs of practical application. The conditions and the range of its validity are investigated, its wider possibilities are explored, and its relation to other methods examined. In the third, the *application* phase, the extended notation or the perfected method becomes once more an instrument for the solution of problems, and is found to be applicable successfully over a field often many times wider than the area in which it originated. Any one of these problems may, in turn, become the starting point of a new cycle.

§ 2. *A historical illustration.* To the general truth of this account the history of mathematics bears abundant witness, but a single illustration must suffice here. The integral calculus originated in the attempt of John Wallis (*Arithmetica Infinitorum*, 1655[2]) to find a general rule for the area (or volume) of a figure in which successive ordinates (or successive cross-sections) could be thought of as proportional to the first, second, third, etc., powers of the natural numbers. Generalising from a few instances he concluded that if the ordinates (or cross-sections) followed the law x^{n-1}, the areas (or volumes) were given by the law $\frac{1}{n}x^n$.[3] So far the heuristic phase. He then inquired whether his rule could be extended to less simple cases—for example, when the ordinates are proportional to the roots of the natural numbers, or inversely proportional to their powers, or follow a still more complicated law. A notable consequence of this inquiry was the invention of fractional and negative indices, by the introduction of which the desired extension of the rule proved possible. Thus, at the end of the second or formal phase, the expressions x^{n-1} and $\frac{1}{n}x^n$ had acquired a greatly widened significance, while the connexion between them expressed by Wallis's rule was still preserved. Armed with the perfected weapon, Wallis, in the application phase, attacked a range of problems which the method in its original form would have been quite unable to solve. Of these the most notable was his famous evaluation of π. Finally, we have a particularly happy illustration of the last remark of § 1. For out of Wallis's investigation of π arose Newton's formulation of the Binomial Theorem.

§ 3. *The heuristic phase in teaching.* Speaking broadly, the three-phase rhythm just described underlies the treatment of algebra to be recommended

[1] The substance of the two introductory lectures of a course on the teaching of algebra.

[2] The present writer has given an account of this book in the *Mathematical Gazette* for December, 1910, and January, 1911.

[3] It must be remembered that at this time only integral indices were used.

in detail in the subsequent lectures. The identity $a^2-b^2=(a+b)(a-b)$ affords a simple instance. It is probably best introduced as a means of simplifying the evaluation of any area consisting in the difference between two squares (Heuristic phase). Analysis should then set to work on the formula and show that it holds good of numbers generally, apart from the question whether they represent linear measurements or not. At the same time, various complications of the identity should be explored—*e.g.* $a^2-(a-b)^2=(2a-b)b$ (Formal phase). The work concludes with examples in which the widened powers of the new instrument are put to use—as in the theory of the maximum and minimum values of quadratic functions, and the solution of quadratic equations (Application phase). The illustration is typical. It will be argued that all the notations and processes commonly taught in the school course should be introduced in a similar way. That is, that they should be presented as instruments for the treatment each of a special type of problems.

It is in connexion with the heuristic phase that the study of the history of mathematics may often be useful to the teacher. For history shows what was the particular need which a given mathematical process was originally designed to meet, and it exhibits the method in its primitive form. With regard to the former point, the teacher may not always—or indeed generally—reproduce exactly in his course the circumstances of the origin of the method, but he cannot fail to find useful guidance in a knowledge of them. With regard to the latter point, it is a great advantage in teaching to present the essentials of a method unobscured by the complications which the "formal phase" has brought into it. Finally, history always illuminates (though it may not determine) the important question of the proper order of introduction of topics and methods. The treatment of the calculus hereafter to be recommended will illustrate all these points. (i) The subject is brought back to its original starting point in history—the evaluation of areas, volumes, etc. (ii) It is developed largely without the aid of the technical notation that was introduced at a later stage of its history. (iii) Its introduction precedes the study of topics (such as the logarithmic and binomial theorems), which in history came after and depended upon the early development of the calculus.

The problems of the heuristic phase may be of any type so long as they fulfil the double condition of being within the circle of the pupil's interests and of offering a suitable starting point for the development of the new method. But in many cases (*e.g.* the notation and methods of elementary trigonometry taught in connexion with surveying) these conditions are best fulfilled by problems of a practical character. Whenever such problems can be found, the search for the methods of solution proper to them acquires the highest zest and the greatest educational value. For even the normally torpid boy will often display unsuspected intellectual fertility when he has to deal with an interesting practical situation, and will rise to astonishing heights of rational criticism of proposals for dealing with it.

The practice of withholding practical applications until the boy has mastered his method in the class-room neglects the motive power that lies in the practical situation. It is based on the schoolmaster's favourite fallacy that "you can't do a thing until you know how to do it." This fallacy reverses the natural order in which genuine intellectual activity proceeds, by placing the formal before the heuristic phase.[1] It forgets also that finding how to do the thing is often a much more valuable process than doing it; the chase

[1] A too familiar instance has just been given me by a student as a piece of his personal experience when a schoolboy. His class in woodwork were taught a new joint which was subsequently to be used in making an Oxford frame. All did the work so imperfectly that they were not allowed to proceed to the frame. A second attempt produced equally unsatisfactory results. In despair the teacher gave them the frame to make—and almost every boy turned out a creditable article!

is better than the quarry. Nevertheless there is a sound principle involved in the practice in question. The schoolmaster sees that on its practical side a problem may be so engrossing or so complicated as to make the boy incapable of attending properly to its mathematical aspect. The reality of this danger is, however, no reason for neglecting a method intrinsically sound. It should rather be a motive prompting a search for the problems which are best suited to be the starting points of each mathematical enterprise. The practical problems which are not adapted to this function should be reserved for the application stage, to be treated when the principle in question has been mastered.[1]

§ 4. *The manipulation of symbolism.* What is here described as the formal phase constitutes practically the whole of the subject when the course has a logical instead of a utilitarian basis. It consists (*a*) in the elaboration of the new method upon its technical side, the side which consists in the manipulation of the symbolism; and (*b*) in the examination of the foundations upon which the method rests, and of the possibilities of usefully extending them.

Both these elements of the formal phase offer material for discussion, and the former is the focus of much current controversy. There is no need to bring that controversy into these lectures. If our general theory is well founded, it should make for reconciliation between disputing schools. For it should give due place to all elements of instruction in which experienced teachers have found value.

Let it be noted, then, on the one hand, that, according to §§ 1 and 2, the formal phase has a vital place in the cycle of progress in algebra; and, on the other hand, that its place is a subordinate one in the sense that manipulative skill is not an independent element to be cultivated for its own sake, but has a definite function in relation to a larger whole. Without a certain degree of facility in manipulation, progress in algebra is impossible. Moreover, that facility must be given by exercises specially designed for the purpose: the conditions of heuristic work cannot be relied upon to make drill superfluous. In including these convictions the tradition of the classroom squares with our theory. On the other hand that theory assigns a definite aim, and therefore a definite limit, to the cultivation of the technical element. Manipulative work arises in the requirements of real problems, and should be a discipline looking forward to the wider needs of yet other real problems. Its cultivation should, in fact, be comparable with "practice at the nets" in cricket. This is a "formal phase" essential to progress in the game. It has the character of a drill; for the professional bowler "sends down" a certain type of delivery time after time until the young batsman has mastered its peculiarities. But the "disciplinary value" of the process consists entirely in the fact that it arises out of the requirements of actual cricket matches, and looks forward to others of yet greater dignity and still more critical importance. A small amount of attention to "fancy strokes" is not illegitimate. For not only may the skilful batsman be permitted the delight of an occasional display of his virtuosity, but he sometimes gains in this way an added power over the strokes which have practical value. But too great a development of a talent in such directions points to the triumphs rather of the circus ring than of the cricket field. The reader may prefer to substitute pianoforte playing for cricket and "studies" for practice at the nets. The parallel would be equally complete.

There is, perhaps, only one other point that calls for attention here. Many teachers who would accept the foregoing limitations of manipulative work for the average boy are afraid that they do not leave room for a command of symbolism adequate to the requirements of the specialist. The study of the development of knowledge alike in the race and in the

[1] Reference may be made to the chapter on this subject (Ch. III.) in the *Report of the L.C.C. Conference on the Teaching of Arithmetic* (1906-8).

individual suggests a more optimistic view. It shows that nothing is really gained by cultivating gymnastic skill which has not *at the time* a value in use; and that the occasion which supplies the demand for a given technical process is the best occasion for its cultivation. This contention may be illustrated by the case of trigonometrical identities.[1] At the stage when they are commonly introduced their use is *nil*; they are relevant to no problem visible above the boy's mental horizon. At best, therefore, he finds in them only the satisfaction that attends even worthless efforts provided they are successful. At worst they are wholly flat and stale as well as unprofitable. Without hesitation, therefore, they should be removed to a place in the course where they have a definite function to fulfil. When in dynamics or the calculus trigonometrical transformations are needed, then skill in such transformations should be cultivated. Such a postponement accords best with the demands alike of economy, of pedagogical reason, and ultimate efficiency.

§ 5. *The logical aspect of the formal phase.* The reproach is often brought against "practical" methods in mathematics that they encourage a habit of loose reasoning. There is reason to fear that the accusation is founded on fact. In accordance with universal precedent the reaction against the excessive and restrictive formalism of the older teaching has expressed itself here and there in a violent effervescence of empiricism. The cure is not a counter-revolution, but reasonable modifications in the constitution—modifications based on a true understanding of the situation. The theory of the former lecture offers a key to its interpretation. We there saw that mathematical activity, while essentially one throughout its history, shows, like all growing things, phases of development directed at different stages by different forces. Practical utility is the ideal at one stage; completeness of logical system at another. Now reasoning must be the instrument of progress at both stages, but it is an instrument applied to different purposes. Reason is always the critic, the referee. At the systematising stage it exercises its functions by bringing unsuspected assumptions to light, and by securing that the logical structure is architecturally sound. At the utilitarian stage its business is to test arguments and methods with a view to their trustworthiness as practical instruments. Its modes of procedure may be diverse in the two cases, but the issue is only obscured by describing the diversity as a difference in accuracy. An engineer who is building an aeroplane (for himself) is not likely to argue less "accurately" than a *savant* preparing a paper for the London Mathematical Society. But reason, the critic, having to judge with reference to two different standards in the two cases, may very well accept a difference in method.

Thus at the utilitarian stage it may often accept the use of what is generally called "induction"—*i.e.* generalisation based on a number of instances and having probable truth only. It will also accept "intuitional" arguments—that is, arguments based upon wide but unanalysed experience,[2] or upon analogy felt to be complete enough though its exact grounds are not exposed. Finally, and as a special case of intuition, it will accept arguments based upon continuity.[3] For all these methods may in special

[1] Accumulating experience is demonstrating its soundness in the parallel cases of the teaching of reading, of number, of needlework, etc.

[2] *E.g.* Euclid I. 4.

[3] A simple instance from geometry: Let APQ be an angle in a semicircle, AQ being the diameter. Let P approach Q. Then as the angle A grows smaller the angle Q grows nearer to a right angle. *When A is zero Q will be a right angle*; *i.e.* the tangent PQ is at right angles to the diameter. From the purely logical standpoint this argument is quite insufficient. For the italicised sentence contains the assumption that a truth (the angle $Q=90°-A$) which holds when A, P, Q form a triangle continues to hold good when they are no longer a triangle. From the utilitarian standpoint it could, however, be accepted. No one could harbour any doubt that the fact is so, nor hesitate to use it for the most important practical purposes.

cases lead to results so certain that no one need hesitate to make practical use of them. "Practical mathematics" and the "new geometry" must not be accused of logical malpractices merely because they employ such methods. They are culpable only if they do not use them in complete accordance with the utilitarian ideal. The engineer would be culpable who did not test adequately the principles upon which he proposes to build a bridge—adequately, that is, from the point of view of the safety of the public. Judged with reference to a totally different ideal Euclid was culpable because he did not remove the "intuitional" element from Prop. 4.[1] We have here a difference not so much of the degree as of the kind of certainty. The common view that a logical system has greater certainty because it depends upon axioms, is not held by those who have most business in these deep waters. "In mathematics," say Messrs. Whitehead and Russell,[2] "the greatest degree of self-evidence is usually not to be found quite at the beginning, but at some later point; hence the early deductions, until they reach this point, give reasons rather for believing the premisses because true consequences follow from them, than for believing the consequences because they follow from the premisses."

The most secure truths derive their certainty either (like Prop. 4) from their congruence with wide experience, or (like the proposition about the angle in a semicircle) from their coherence with other true propositions. It is not sufficiently realised that the logical stage in mathematics is a kind of intellectual game. The principles of the game are to make explicit all the assumptions implicit in your arguments, to reduce them to as small a number as possible—your "axioms"—and to reconstruct your reasonings in the form of a system depending entirely upon these.

This "game" is also, no doubt, one of the noblest works that man can put his hand too, but the motive which urges him to it is rarely strong in the schoolboy. This is not an assertion that the schoolboy is an unreasoning animal. On the contrary his logical powers show themselves vigorous enough in practical matters. There is no reason why they should not receive from practical occasions abundant opportunities for exercise and training. That is to say, there is no reason why reasoning upon the utilitarian level should not be cautious, thorough and penetrating. A boy may make use of induction—but he should recognise the risks attending it and the need of manifold verification. He may make an *ad hoc* assumption based upon analogy or experience—but he should know that he is doing so. Upon these conditions the utilitarian motive which appeals to him will guarantee a far more real logical training than the systematising motive which is unnatural to him.

Recognition of the independent rights of the utilitarian stage instead of depressing should actually raise the general level of school reasoning. For it implies bringing the "practical methods" of the laboratory and the theoretical work of the class-room under the same canons of criticism. The present divorce between them is good neither for practice nor for theory. Practical methods pursued uncritically degenerate into crude empiricism; while theory, having nothing to say about methods which have practical utility, comes to be regarded as superfluous and a nuisance.

To avoid misunderstanding it should, perhaps, be added that there is no contention that upon the utilitarian level arguments based upon induction, analogy, and continuity are to be preferred. On the contrary the ideal of practical efficiency is better served the more completely our arguments are based upon analysis, and the more they are shown to cohere with other arguments. What it is important to claim is that arguments of the lower

[1] Or else, like Hilbert (*Grundlagen der Geometrie*), place the proposition among his axioms.

[2] *Principia Mathematica*, Preface.

type have their place in mathematics, and that the development from the lower to the higher should be a transition naturally made by a mind seeking greater efficiency and economy in its thinking.[1] T. P. NUNN.

THE NATURE OF ELEMENTARY MATHEMATICS FROM THE MODERN STANDPOINT.

THE character of the modern treatment of elementary mathematics is such that mathematics might reasonably be described as the Science of Classification. The mathematician may be supposed to begin as a collector, whether of butterflies, minerals, or even of cigarette cards, does not matter. Let us consider that he has collected so many different kinds of things as to become virtually a collector of collections, each collection having its own classification, and that his next step is to compare these classifications with each other, that is, to become a classifier of classifications. The mere comparison of classes requires the idea of cardinal number, since the only thing which can be said about two classes, irrespective of the nature of their individual members, is the number of individuals in each class. But the comparison of classifications which do not consist merely of single classes introduces ideas which are essential to the further development of the idea of number, and to the development of Geometry, and which should therefore precede that development. It is in this super-classification or comparative study of classifications that the foundations of mathematics are to be sought, and the question therefore arises, whether the presentation of the rudiments of mathematics should not be modified so as to recognize this fact. Of the fundamental ideas of mathematics, four especially, namely those of correspondence, order, group, and of multiple correspondence, seem to lend themselves, in their elementary treatment at any rate, to the method in which they are regarded as belonging to the Science of Classification, and will be considered in that light here, the last three being regarded as developments of the first.

Needless to say, this description of the evolution of mathematics does not agree either with the history of mathematics, as commonly given, or with the stages of mathematical instruction in schools. It must be observed, however, that the historian of mathematics is influenced by his views as to the nature of mathematics, and that the appearance of works like J. W. Young's *Fundamental Concepts of Algebra and Geometry*, suggests that even the traditional course may be about to undergo changes. It is true that the author of that book considers that the modern treatment of elementary mathematics demands too high powers of abstraction to be employable in schools in the form in which it is usually presented, namely, as a system of deductions from a comparatively small number of unproved axioms or assumptions about undefined terms. This treatment is of such a generalized character that both ordinary algebra and ordinary geometry may be among the implications of the same set of axioms, so that the two subjects may be said to have become fused into one.

Now, while admitting that it is not necessary in school work that the number of undefined terms and unproved statements should be reduced to a minimum, yet, since the ideas now accepted as being really fundamental can be understood by practical exercises in the same way as those which are not fundamental, there seems no reason for the almost exclusive attention paid to the latter in schools. The learner at present begins with the construction

[1] See also Section VI. of the article on Wallis in the *Mathematical Gazette* for February, 1911.

and measurement of diagrams, with paper folding, and other practical exercises familiar to teachers. These exercises yield experimental facts, and indicate the kind of implications which it is desirable that the axioms of geometry should contain. But to learn the names and properties of the more important types of classifications presents no harder task, since it too can be accomplished with the aid of practical exercises in classification, as well as of those numerous illustrations which both school and everyday life afford, quite apart from what is commonly regarded as mathematics. Models of a duplex, triplex, and quadruplex can be made from pieces of cardboard properly coloured and shaped, and by similar models can be understood the true meaning of the all-important terms Correspondence and Order. For a correspondence, or function, or correlation, may be explained as a classification and cross-classification of the things which correspond. And again, since when like things appear on both sides of a correspondence they are in general arranged as it were in a chain, of which the correspondence supplies the links, it is also to classification and cross-classification that we turn for an explanation of the idea of order.

Among the common examples of a duplex may be mentioned the classification of nouns by declension and case, that of simple salts by metal and acid radical, that of cards in a pack by suit and value. Among the examples of functions, the correspondence of English and French words in an English-French Dictionary, of parents to children, of subject and object, of a transitive verb. These illustrations and the use of models would, if necessary, afford sufficient material by themselves for the introduction of terms like variable, direct and inverse, product, power, and others, and for the symbolic statement of a correspondence between two variables, without reference either to arithmetic or to geometry in the common sense. Indeed, it is logically necessary to consider the meaning of multiplex and of order separately before combining them to consider an ordered multiplex or "space" of which the classified things form the "points." A pupil who acquired his geometrical ideas in this order would naturally look upon geometrical instruments like set-square and metre rule, as playing a part similar to that of scissors and paint brush in constructing the model of a multiplex.

The third important idea in elementary mathematics, that of a Group, links itself to that of correspondence in the fact that a Group may be looked on as a correspondence not of things to other things, but of things to functions. A correspondence of things to functions need not be a Group, but every Group can be regarded as an instance of such a correspondence, and therefore as arising from a classification and cross-classification.

In the case of a Group, the variable stands for things when one side of the correspondence is in question, and for functions when the other side is in question, but in both instances, and in all the examples above, it stands for any member of a simple class. Now the peculiarity of multiple correspondence, which we take as the fourth important idea, and which is the foundation of Multiple Algebra, lies in the fact that the variable stands not for a member of a simple class, but for a member of a duplex, triplex or other multiplex. On either side of the correspondence we have not a simple class but a multiplex. The symbolic notation for a multiple correspondence must therefore depend on the one hand on that adopted for representing the position of a thing in a multiplex, and, on the other hand, on that adopted for representing correspondence. And, likewise, the extent to which the idea may be introduced into elementary work is limited by the need for a clear understanding of the meaning both of a multiplex, and of a correspondence. It remains for teachers to discover how far the use of models and of simple illustrations is adequate to furnish these ideas. C. ELLIOTT.

MATHEMATICAL NOTES.

362. [K. b. j.] *Locus of orthocentre.*

Given the base BC and the vertical angle A, the locus of the orthocentre H is the arc of a segment of a circle, described on a line equal to the given base and at a fixed distance from it, containing an angle equal to A. It is frequently stated that this angle is not A, but the supplement of A. The statement is erroneous, being incomplete when A is acute, and only partially true when A is obtuse. It leads to similar errors in the case of the locus of the nine-point circle of such triangles. CHARLOTTE SMITH.

363. [K : 6.] The following simple method of determining (in three-dimensional co-ordinate geometry) the length of the perpendicular from a given point to a given line and the co-ordinates of its foot does not seem to be mentioned in the text-books.

Let the point be $P(1, 2, -1)$ and the line $(x-1)/3=(y-4)/2=(z-2)/5$.

Any point on this line may be represented by

$$(1+3t,\ 4+2t,\ 2+5t).$$

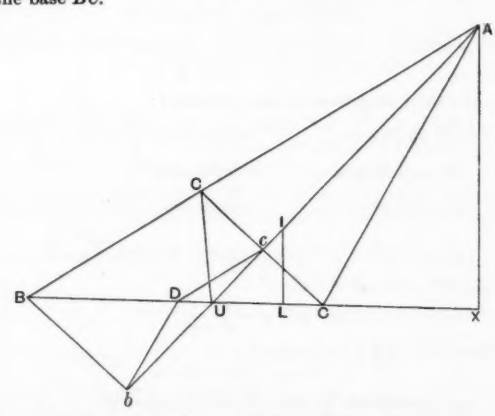
The square of the distance of P from this point is

$$\begin{aligned} 9t^2+(2+2t)^2+(3+5t)^2&=(38t^2+38t+13)\\ &=38(t+\tfrac{1}{2})^2+3\tfrac{1}{2}. \end{aligned}$$

But the perpendicular is the shortest line of this class, and the above expression is clearly least when $t=-\frac{1}{2}$, giving $3\frac{1}{2}$ as the least value.

Hence the length of the perpendicular is $\sqrt{3\frac{1}{2}}=1{\cdot}87$, and its foot is $(-\frac{1}{2}, 3, -\frac{1}{2})$, putting $t=-\frac{1}{2}$ in $(1+3t,\ 4+2t,\ 2+5t)$. H. PIAGGIO.

364. [K. 2. c.] ABC is a triangle, I its inscribed circle, and AI bisecting the angle A meets the base BC in U. From B, C perpendiculars Bb, Cc are drawn on ICU; Cc meets AB, in C', such that $AC=AC'$; D is the middle point of the base BC.



Then, as in CASEY'S SEQUEL, $Db=Dc=\frac{1}{2}(AB-AC)=DI$, where I is the point of contact of the inscribed circle with BC.

The circle Σ whose centre is at D and whose radius $=\frac{1}{2}(AB-AC)$ cuts orthogonally the inscribed circle; and $Ib.Ic=r^2$.

Since $bU.Uc=DC^2-DU^2=DI^2-DU^2=DU.DX-DU^2=DU.UX$, therefore U is a point on the radical axis of Σ and Ω (the Nine-Point circle whose

centre is N). Also, since $C'UB = C - B$, it may be shewn that UC is perpendicular to the join of the centres of Σ, Ω.

Thus UC' is the radical axis of Σ, Ω.

Now, difference of squares of the tangents from any point to Σ, Ω is twice rectangle contained by the perpendicular from that point on UC' and the distance between the centres of Σ, Ω ($=\frac{1}{2}R$).

Employing the theorem for the point I,

$$r^2 - \left(NI^2 - \frac{R^2}{4}\right) = Rr,$$

for perpendiculars from I on UC', UC are equal.

Hence $r^2 + \frac{1}{4}R^2 - Rr = NI^2$, and $NI = \frac{1}{2}R - r$. R. F. DAVIS.

365. [K. 8. b.] *To calculate the sides of a cyclic quadrilateral having given its diagonals, its area, and the radius of the circumscribing circle.*

Let e, f, be the diagonals, Q the area, and R the circum-radius. Denote the sides by x, y, z, w; then, since the opposite angles are supplementary,

$$e^2 = \frac{(xz+yw)(xw+yz)}{(xy+zw)} \quad \text{and} \quad f^2 = \frac{(xz+yw)(xy+zw)}{(xw+yz)};$$

hence $e^2 f^2 = (xz+yw)^2$ or $ef = xz+yw$,

the well-known theorem of Ptolemy.

Again, $$Q = \frac{xye}{4R} + \frac{zwe}{4R} = \frac{e}{4R}(xy+zw);$$

$$\therefore \; xy+zw = \frac{4RQ}{e}.$$

Similarly, $$yz+xw = \frac{4RQ}{f}.$$

Lastly, $$16Q^2 = 4(xw+yz)^2 - (x^2-y^2-z^2+w^2)^2$$

$$= \frac{64R^2Q^2}{f^2} - (x^2-y^2-z^2+w^2)^2;$$

$$\therefore \; x^2-y^2-z^2+w^2 = 4Q\sqrt{\frac{4R^2}{f^2}-1}.$$

Therefore the sides are given by the equations:

(1) $xy+zw=m$, where $m = \dfrac{4RQ}{e}$;

(2) $xz+wy=n$, where $n = ef$;

(3) $yz+xw=r$, where $r = \dfrac{4RQ}{f}$;

(4) $x^2-y^2-z^2+w^2=s$, where $s = 4Q\sqrt{\dfrac{4R^2}{f^2}-1}$.

Now $x^2z^2 + w^2y^2 - y^2z^2 - x^2w^2 = n^2 - r^2$

or $$x^2-y^2 = \frac{n^2-r^2}{z^2-w^2};$$

substitute this in (4) and put $z^2 - w^2 = p$;

$$\therefore \; p^2 + ps - (n^2-r^2) = 0,$$

giving $$p = \tfrac{1}{2}\{-s \pm \sqrt{s^2+4(n^2-r^2)}\}.$$

Similarly, $g = \frac{1}{2}\{s \pm \sqrt{s^2+4(m^2-r^2)}\}$, where $g = x^2 - z^2$.

Also, $$(x+w)^2 - (y-z)^2 = s+2r, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

and $$(x-w)(y-z) = m-n$$

or $$y-z = \frac{m-n}{x-w}.$$

Then $(x^2-w^2)^2-(m-n)^2=(s+2r)(x-w)^2$.

But $x^2-w^2=x^2-z^2+z^2-w^2=g+p$;

$$\therefore\ x-w=\pm\sqrt{\frac{(g+p+m-n)(g+p-m+n)}{s+2r}},$$

and $$x+w=\frac{x^2-w^2}{x-w}=\pm(g+p)\sqrt{\frac{s+2r}{(g+p+m-n)(g+p-m+n)}};$$

$$\therefore\ 2x=\frac{(g+p+m-n)(g+p-m+n)+(g+p)(s+2r)}{\sqrt{(g+p+m-n)(g+p-m+n)(s+2r)}};$$

hence, w and z may be found from $x^2-z^2=g$, and $z^2-w^2=p$; and then y from (1).

Particular cases.

(1) When $e=f=2R=a\sqrt{2}$; $Q=a^2$.

Then $m=n=r=2a^2$, $p=0$, giving $p=g=0$.

By substitution in the above expression for x, we get the indeterminate form $\frac{0}{0}$. Since, however, $p=g=0$, $\therefore\ x=z=w$, and from (4) $x=y=z=w$, s being zero.

$\therefore$ from (1) $x=a$, and the figure is a square of side a.

(2) When $e=f=2R=17$; $Q=120$.

Then $m=r=240$, $n=289$, $s=0$, giving $p=161$, and $g=0$; hence,

$$x=15=z \quad \text{and} \quad y=w=8,$$

and the figure is a rectangle.

(3) When $e=25$, $f=23{\cdot}4$, $R=12{\cdot}5$, $Q=234$, then $m=468$, $n=585$, $r=500$, $s=352$, giving $p=175$, and $g=176$; hence, we find

$$x=24,\ y=7,\ z=20,\ w=15.$$

N.B. Only the positive values of the radicals have here been considered.

F. G. W. Brown.

366. [L¹. 4.] *Power of a Point with respect to a Conic.*

(1) If tangents TP, TQ are drawn from $T(X, Y)$ to an ellipse, centre C, the area of the quadrilateral $CPTQ$, according to an example in Smith's Conics, is $(a^2Y^2+b^2X^2-a^2b^2)^{\frac{1}{2}}$. Similarly for a hyperbola.

(2) The power of S with respect to the eccentric circle of T is also readily found to vary as the power of T with respect to the conic. A. C. Dixon.

367. [E. 5.] *Proof that* $\int_0^\infty \frac{\sin x}{x}\,dx=\frac{\pi}{2}$.

(1) If $\int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)x}{\sin x}\,dx=u_n$, and n is a positive integer, $u_n-u_{n-1}=0$, and thus $u_n=u_1=\frac{1}{2}\pi=\frac{1}{2}(u_n+u_{n-1})=\int_0^{\frac{\pi}{2}} \sin 2nx\cot x\,dx$.

(2) Hence $\int_0^{n\pi} \frac{\sin x}{x}\,dx-\frac{\pi}{2}=\int_0^{\frac{\pi}{2}} \frac{\sin 2nx}{x}\,dx-\int_0^{\frac{\pi}{2}} \sin 2nx\cot x\,dx$

$$=\int_0^{\frac{\pi}{2}} \sin 2nx\left(\frac{1}{x}-\cot x\right)dx$$

$$=\frac{1}{2n}\cos 2nx\left(\cot x-\frac{1}{x}\right)\Big|_0^{\frac{\pi}{2}}+\frac{1}{2n}\int_0^{\frac{\pi}{2}} \cos 2nx\left(\operatorname{cosec}^2 x-\frac{1}{x^2}\right)dx.$$

In this expression the terms at the limits contribute $\pm\frac{1}{n\pi}$; since $\operatorname{cosec}^2 x - \frac{1}{x^2}$ is positive, the last term is numerically less than

$$\frac{1}{2n}\int_0^{\frac{\pi}{2}}\left(\operatorname{cosec}^2 x - \frac{1}{x^2}\right)dx \quad \text{or} \quad \frac{1}{n\pi}.$$

Hence $\int_0^{n\pi}\frac{\sin x}{x}\,dx - \frac{\pi}{2}$ is between $\pm\frac{2}{n\pi}$.

(3) If $n\pi < a < (n+1)\pi$, $\int_0^a \frac{\sin x}{x}\,dx$ lies between

$$\int_0^{n\pi}\frac{\sin x}{x}\,dx \text{ and } \int_0^{(n+1)\pi}\frac{\sin x}{x}\,dx.$$

Hence $\int_0^a \frac{\sin x}{x}\,dx \to \frac{1}{2}\pi$ when $a \to +\infty$. A. C. Dixon.

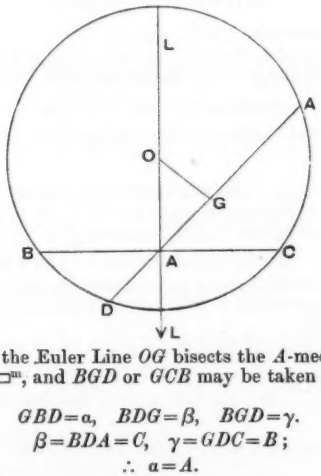
368. [K. b. β.] *Median triangles.*

1. The Median Triangle of ABC has its sides equal to GA, GB, GC. Given the base BC, and the vertical angle, it is required to describe a triangle which shall be similar to its own Median T.

Bisect BC in A': with centre A' and radius $= a\sin bc^2$, cut the circumcircle in A.

Then $\frac{1}{4}.a^2 =$ power of $A' = \frac{a\sqrt{3}}{2}.A'D$.

$$\therefore\ A'D = \tfrac{1}{3}.A'A = A'G;$$



$\therefore\ DG = AG$, and the Euler Line OG bisects the A-median.

Also, $BGCD$ is a $\square^m$, and BGD or GCB may be taken as the Median T. of ABC.

Let $\quad GBD = \alpha, \quad BDG = \beta, \quad BGD = \gamma.$

Then $\quad \beta = BDA = C, \quad \gamma = GDC = B;$

$$\therefore\ \alpha = A.$$

Thus ABC is similar to its own Median T.

2. Since $A'G.A'A = \frac{1}{4}a^2 = A'B^2$ or $A'C^2$,
it follows that the circles BGA, CGA touch BC at B and C.

Since $$\frac{3}{4}.a^2 = A'A^2 = \frac{2b^2+2c^2-a^2}{4},$$

$$\therefore\ b^2+c^2 = 2a^2;$$

$$\therefore\ GA^2 : GB^2 : GC^2 = 2b^2+2c^2-a^2 : 2c^2+2a^2-b^2 : 2a^2+2b^2-c^2$$
$$= a^2 : c^2 : b^2 ;$$

$$\therefore\ A=\alpha,\quad B=\gamma,\quad C=\beta, \text{ as before.}$$

(Compare Emmerich's *Brocardschen Gebilde*, p. 17.)

3. Neuberg Circles.

On OA' take $A'L=A'L'=A'A$; and describe a (Neuberg) circle, having L, L' for limiting points.

Draw any secant $A'PQ$ to this circle.

Then each of the triangles PBC, QBC is similar to the Median T. of the other.

W. GALLATLY.

REVIEWS.

An Elementary Treatise on Cross-Ratio Geometry, with Historical Notes. Rev. J. J. MILNE, M.A. Pp. 288. Cambridge. 1911.

Mr. Milne is one of the few English mathematicians who are also competent classical scholars, and he has made use of his classical knowledge to work through a great deal of the old Greek Geometry. He found that the Greek geometers, notably Apollonius and Pappus, besides knowing the cross-ratio property of pencils of lines, had solved many problems involving coaxial homographic ranges, treatises having been written by them on the subject. Apollonius also had given theorems intimately connected with the cross-ratio properties of points and tangents of a conic. Their methods were heavy and difficult, and Mr. Milne shows how the various great European geometers who revived and extended the ancient geometry gradually evolved the present powerful notation which systematises and simplifies the methods of solving the old problems so immensely, and has led to the present knowledge of this most fascinating subject.* Among the most interesting parts of the book in its historical aspect are Chapter VIII., dealing with three books of problems written by Apollonius, and Appendix I., giving the enunciations of Pappus' Lemmas on Euclid's porisms, with modern solutions of them. But the book is full of historical references, not only to the Greeks, but also to Pascal, Chasles, and other great Continental geometricians and to those of this country who have contributed notable theorems on this subject.

The book leads the student carefully from the rudiments to the more difficult theorems, weaving in the historical references and extracts as it proceeds, and is eminently readable. Its historical plan has led the author to devote about half the book to linear theorems and problems, which are treated with unusual fullness. Then he proceeds to properties of conics and systems of conics. Special mention should be made of the part dealing with contra-polar or harmotomic conics, which are homographic generalisations of orthogonal circles, as is pointed out in his last chapter, where a whole series of properties of circles and systems of circles are thus generalised. Except in this last chapter, and in one or two other articles, no conic properties are derived from the circles, but are self-contained.

The author has adopted this method chiefly because it is in accordance with the spirit of the ancient geometers, but also because he felt that this method is more comprehensive and satisfactory and convincing. He has, however, in some cases suggested that the student may, if he prefers, derive the conic theorems by direct projection from the circle, which, for examination purposes, might be occasionally quicker and simpler. But the author has very carefully stated and proved the fundamental conic theorems on which he bases his method, and they bring out very forcibly the peculiar infinitesimal-circle nature of conic foci.

*A study of Appendix II., where Pascal's theorem is proved by the method of Euclid and Apollonius, serves as an illustration of how much has been gained in this respect. It is a most ingenious proof, but is rather staggering to read, and an excellent 'contrast to the simple proof given in the text.

There is a very complete table of contents and a valuable index to facilitate reference, and at the end is a number of blank pages for a student's notes and solutions—a somewhat unique and very useful feature in a text-book, which, we think, might be imitated by other authors.

We ought not to conclude without stating that where the solution of a problem requires the finding of the common points of two coaxial homographic ranges the author in his worked-out examples, which are numerous, has actually constructed these points and completed the solution by a new method which is very simple and leads to most accurate results, and which he explains very fully. The same method enables him to construct pairs of corresponding points with great facility, and is of special interest as, though not specifically mentioned by the author, it is applicable to the construction of geometrical images in systems of lenses, which, indeed, are a particular case of coaxial homographic ranges.

The diagrams throughout the book are most accurately drawn and are models of what such diagrams should be, and in Chap. XVI. we find the ideal chords and common self-conjugate triangle drawn in all the cases where two conics do not intersect in four real points.

We prophesy a great future for this most interesting and valuable book. It is suitable for scholarships and university purposes, and is full of interest for the expert, being not only a text-book but a historical survey by a man who is unusually well equipped for the task and has evidently found it a labour of love. We wish it all success. L. A.

La Méthode dans la Philosophie des Mathématiques. By MAXIMILIEN WINTER. Pp. iii+200. 1911. (Alcan.)

M. Winter's problem is: What is the method which, at the present time, offers scientific guarantees sufficient for the critical examination of the foundations of mathematics?

The book is divided into three parts: (1) The metaphysical methods (pp. 1-48) must, says the author, be dismissed. In the metaphysics of science the primitive ideas are the vague notions of common sense. (2) On the attempts to determine the laws of mathematical thought by *logistics* (mathematical logic, pp. 49-101). Regarding logistics as a *calculus*, M. Winter tries to map out the "restricted but real domain of the science founded by Boole and Schröder" (p. 11). The author's opinion of logistics is moderately favourable, but he considers that the essential of mathematical thought has not been absorbed into it, and it throws no light on certain mathematical questions. (3) M. Winter then applies to two theories (the theory of numbers since Gauss, and the genesis and development of the theory of the resolution of algebraic equations) the historical and critical method used by Mach, "the only method which seems capable of giving interesting results" (p. ii).

In general, one can say of this book what Dr. Venn said of another French book, that it "possesses the national merit of lively and transparently clear exposition of all that is understood." M. Winter, for Mr. Russell's work, quotes Couturat, and so makes the somewhat natural mistake, owing to Couturat's wording, of considering Russell to be a formalist, as Professor Natorp does. M. Winter does not recognise (p. 97) the fundamental character of the paradoxes of the theory of aggregates.

Theoretische Arithmetik. Von OTTO STOLZ und J. ANTON GMEINER. I. **Abteilung: Allgemeines. Die Lehre von den rationalen Zahlen.** Zweite Auflage, umgearbeitet von Dr. J. ANTON GMEINER. (Dritte Auflage der Abschnitte I.-IV. des I. Teiles der **Vorlesungen über allgemeine Arithmetik**, von O. STOLZ.) Pp. vi+148. Bd. M. 5,20. 1911. (Teubner.)

This is the second edition of the first part of the fourth volume of Teubner's collection of text-books on mathematics, issued in connection with the *Encyklopädie der mathematischen Wissenschaften*, and—what is an unusual delight with German books—well bound in strong and neat cloth.

Like the first edition of the work under review, this edition contains four sections, on the concepts of magnitude and natural number, and the analytic and synthetic theories of rational numbers. This edition has been thoroughly revised and worked through: some notes left by the late Professor Stolz have been

incorporated, and Dr. Gmeiner has succeeded in reducing the number of axioms used in the work. Thus, Peano's treatment of the natural numbers is followed except in that Peano has six axioms, whereas Gmeiner has five (p. 15). It might be remarked that "usf." or "..." is a device of mathematicians which appeals to intuition, but often slurs over a want of logic.

The general features of this book are by now well-known. Perhaps it is unavoidable to teach mathematics at the present time as if we neither believed that arithmetic had so many axioms apart from logic as people believed forty years ago, nor that it had none—as we now know. PHILIP E. B. JOURDAIN.

Plane Trigonometry. L. K. GOSH. (Halder, Calcutta.)

This volume is written especially for the Intermediate Examinations in Arts and Science of the Indian Universities. It labours under the disadvantage of bad type and poor printing, this being especially noticeable in the signs +, −, and =, and the spacing of the working of proofs and exercises. The subject is treated in much, perhaps too much, detail; a multitude of special cases only serves to confuse a student. The author is evidently unacquainted, or not in sympathy, with the present idea of gradual extension and generalisation. In our opinion the early introduction of circular measure is not to be recommended, and the obsolete (some say never used) "grade" should have been omitted. The ratios of angles of any magnitude are illogically defined from a triangle of reference instead of from the coordinates of the extremity of the vector; illogically, because the triangle bears a totally different relation to the angle in each quadrant. Much space is wasted on the changes in sign and magnitude of the ratios, which could have been saved by treating this section graphically. Not sufficient regard is paid to the sign of a line, *e.g.* on page 35, PM/AP is given as the sine of A instead of MP/AP, and this error persists through the book. The author seems unaware that 355/113 is a closer approximation to π than 3·14159. The volume would have been more interesting if solution of right-angled triangles could have been introduced earlier.

Advanced Arithmetic. M. H. JURDAK.

This volume is intended for school use in Syria, and is very carefully thought out.

A little too much space is devoted to complex fractions, there being seventeen pages of exercises and problems on this section! Decimals and the Metric system are treated very conscientiously; but what is now deemed the important part, approximation methods of multiplication, division, etc., are apparently left out. It is stated that the Unitary method is used, but the worked-out examples have a strange appearance as examples of this method; it is nowhere stated, or rather insufficiently insisted on, that Interest and Discount are all "examples on Proportion"; in fact these sections precede the chapter on the Unitary Method. Horner's method for cube roots, if any, is preferable to that founded on the formula $300a^2+30ab+b^2$ for the new divisor. At the end of the book, we find the best part, namely, introduction to Algebra by generalisation; this should prove very useful in accustoming the pupil to the use of symbols.

Bearing in mind the standard set up by Pendlebury and others, it is difficult to judge this book justly; no doubt it will prove very serviceable for the special use for which it is intended.

Arithmetic for Schools. F. C. BOON. (Mills & Boon, Ltd.)

This seems to be an exceptionally well-planned book for school use. Every difficulty likely to arise is taken in hand and thoroughly well explained. The importance of rough checks is rightly insisted on throughout. Approximation methods in decimals are accorded particularly full treatment, but the method advised is open to objection. Experience shows that it is shorter and quite as accurate to work to "one figure more" and allow "for the figure to carry" as to work to "two figures more." Again, the Unitary Method is relegated to a second place with regard to the Method of Ratios, but most readers will at once recognise in the Method of Ratio what they have generally known as the abbreviated Unitary Method. Logarithms and their connection with indices are well done, but no attempt, wisely perhaps at this stage, is made to justify the

existence of such an index as 0·3010300. There is a valuable section on limits of error; but the method of basing the theory on formulae to be memorised is open to objection; also we prefer the method of upper and lower limits to that of relative error when teaching young students. The book ends with sections on Graphs, Theory of Numbers, and Problems, and a large collection of harder Miscellaneous Exercises.

Workshop Arithmetic. H. A. DARLING. (Blackie.)

This little book should prove very useful for the class to which it appeals. All kinds of different things have been drawn upon to furnish examples, from engineering to cricket. This makes the book 'live' and interesting. A curious misprint occurs throughout the closing pages, "lateral" for "literal." The explanation of logarithms is rather short, as is also the important section on contracted methods with decimals.

New School Geometry. R. DEAKIN. (Mills & Boon.)

The excellent plan of leading up to a theorem by means of well-planned exercises is adopted in this text-book. The author, however, seems in several cases to be doubtful to which side to give his adherence—the Euclidean or non-Euclidean, for several pairs of alternative proofs are given. This savours of an attempt to please everybody, and is a mistake, especially when the alternative proof, *e.g.* Prop. 9, has to be taken out of the order in which it is put in the book. One of the best portions of the volume is the careful way in which congruent triangles are led up to and explained. In fact the pupil, carefully drilled, should all through be quite convinced of the truth of a theorem before the demonstration is set before him; and this cannot fail to help him to understand the demonstration. Sufficient practical work is given concurrently to interest the pupil; and a selection of 150 riders brings the book to a close.

The printing leaves nothing to be desired, but the pagination, reminding one of the Todhunter of one's youth, is poor. A Theorem seems to gain dignity when it has a page to itself, or at least starts a fresh page; also it is a matter of opinion whether a proof in which abbreviations are barred is anything like so easily assimilated as one which, by the use of abbreviations, can be taken in at a glance.

School Geometry. CHAMPION and LANE. (Rivingtons.)

This book follows a sequence of its own as regards the early propositions, and does not lose in value thereby. The propositions are well set out, with large clear diagrams, abbreviations being freely used from the start, and each proposition starts on a fresh page, giving the book a nice appearance on first opening. We do not meet with any practical work, however, until page 57, and then we get a batch to page 72, in the form of propositions, instead of applications of the preceding fundamental facts. Areas come before the circle, and here the first few propositions of Euclid's Book II. are treated with scant ceremony, and although "quoted freely in the remainder of the book," are dismissed in about half a page. We have always found it one of the most difficult things in teaching to get a student to understand the relation of these propositions to the mensuration ideas of the equivalent algebraical identities. The difficulty is that a student, being told of the relation, insists subconsciously in thinking of AB^2 as $A \times B \times B$ instead of x^2 where x is the length of AB. The scant treatment afforded to this section will hardly improve matters. The circle is satisfactorily handled, but the objection raised to the limit definition, in which the two points defining the secant are supposed to *coincide*, applies also here. If the limit definition is used it must have more careful treatment. The book closes with a section on Ratio, and here again the fundamental algebraic identities are dismissed in half a page. Several of the propositions given in this section seem redundant, as they follow naturally from the fundamental algebraic identities.

The book closes with a large number of sets of miscellaneous riders which are distinctly good.

J. M. CHILD.

Higher Mathematics for Chemical Students. By J. R. PARTINGTON. Pp. 272. 5s. 1911. (Methuen.)

Mr. Partington is to be congratulated on the successful manner in which he has provided for the needs of the chemical student who is anxious to make himself acquainted with the value of mathematics as one of the weapons in his armoury. His little book of about 250 pages covers just the ground that is necessary for the student of physical chemistry who desires to study the mathematical representation of the connection between phenomena and to know something of the methods and processes as far as they are applicable to his own sphere of labour. As far as is possible the student is kept in touch with realities throughout. For instance, the method of integrating partial fractions is illustrated from the equations of chemical kinetics, and the method of mean value from Horstmann's work on dissociation pressures. Ample references are given for those who intend to master the use of the new weapon to which they are here so effectively introduced.

The Elements of Analytical Geometry. By G. A. GIBSON and P. PINKERTON. Pp. xxi+475. 1911. (Macmillan.)

Teachers of the old school would have opened their eyes to find a work on analytical geometry in which the student has to read 312 pages before finding the canonical equations of the conic sections. In this excellent introduction to the principles of the subject the moment the straight line and circle have been mastered we find the conchoid, cissoid, and Witch of Agnesi followed by the conics as instances of easy loci. Then follow graphs of equations, successive approximations, maxima and minima, asymptotes, and the tracing of more difficult curves. This part of the book, as might be expected from Dr. Gibson's share in it, is a revelation in the way of thoroughness. But we cannot help feeling that the authors would have been better advised had they assumed that by the time a boy has reached this stage in his work it is well for him to be taught the value of calculus methods. A chapter is devoted to the ellipse as the orthogonal projection of a circle, and the analytical and geometrical discussions of secants, tangents, and normals are dealt with separately. Pole and Polar theory is approached by way of Joachimsthal's Section equation. Freedom equations are wisely brought in at an early stage. "The unwarranted and even mischievous" separation of geometrical and analytical conics has been kept well in mind, with the result that "the essentials of the older treatises on Geometrical Conics" are included. The supply of examples is large enough to permit of considerable variety of choice on the part of the teacher or the clever boy. There is no attempt to make this book satisfy the requirements of any particular examining body, but we should imagine that any student who has mastered its contents would be well to the front of many whose introduction to the subject has been on more old-fashioned lines. At any rate no teacher of analytical geometry can fail to be struck with the care and thought that the authors have lavished upon their work.

A New Trigonometry for Schools and Colleges. By the REV. J. B. LOCK and J. M. CHILD. Pp. xii+488. 6s. 1911. (Macmillan.)

The scope of this volume is indicated by the statement of the authors that it contains all that is required for candidates for University scholarships, and is suitable as an introduction to the more advanced parts of Prof. Hobson's Trigonometry. It also contains a considerable amount of material which the young teacher will not readily find elsewhere, *e.g.* the sections on isogonal points, Lemoine, Triplicate, and Cosine Circles, etc. The properties of the circumscribable quadrilateral are studied in connection with the four coins theorem: If four coins are placed on a table so that each touches two, and only two, of the others, (i) the quadrilateral determined by their centres is circumscribable, and (ii) the quadrilateral determined by their points of contact is cyclic. For the escribed quadrilateral the four rings are again utilised, with, "if necessary, a break in the circumference of each to allow the rings to cross." Then, as there are four, two, or no external points of contact, the quadrilateral is circumscribable. The latter case is fully treated, requiring eight diagrams. We are not sure that the earlier sections of this volume are entirely suitable for the

tender years at which boys nowadays begin the study of the subject, but as a revisional course they would prove excellent. The descriptions of surveying instruments are done carefully and well, and the illustrations are as good as any we can remember. The geometrical treatment of imaginaries is another good point which must not be overlooked. The examples and test papers seem to be judiciously selected and arranged, and hints for solution of harder questions are given at the end of the book. On p. 360 there is an odd misprint of "triangle" for "circle." The book is beautifully printed.

Physical and Chemical Constants and Some Mathematical Functions. By C. W. C. KAYE and T. H. LABY. Pp. 163. 4s. 6d. net. 1911. (Longmans, Green.)

This collection of up-to-date English physical and chemical tables was begun at the Cavendish Laboratory, and the proof-sheets of the various sections have been read by experts "in the subjects with which their names are associated." In many cases references are given to original papers and books, with the year of publication of results. The authors claim that the collection of data referring to Radioactivity and Gaseous Ionisation is the first of the kind that has appeared. The book is beautifully printed, and is bound in a limp cover. There is little doubt that this useful collection will soon be in great demand.

Elementary Applied Mechanics. By A. MORLEY and W. INCHLEY. Pp. viii+382. 1911. (Longmans.)

Prof. Morley's previous works on Mechanics for Engineers and on Strength of Materials will induce many readers to turn with interest to see what his further experience has taught him with regard to the right method of approach for the beginner who wishes to cover the ground indicated by Stage I. Board of Education. The first point to notice is the realisation of the value of numerical calculations to the beginner. Mr. Inchley, to whom this part of the book has fallen, supplies the student with plenty of carefully worked-out examples and an ample store on which to try his mettle. The figures are clearly drawn, and are over 250 in number. The presentation of the subject follows the usual lines, and the style is simple and clear. There is little doubt that this will be found a satisfactory introduction to the subject for those for whom it is intended.

THE LIBRARY.

THE Librarian begs to acknowledge with thanks the gift by the Rev. J. J. Milne, of 15 works for the Library.

The Library has now a home in the rooms of the Teachers' Guild, 74 Gower Street, W.C. A catalogue will be issued to members in due course, containing the list of books, etc., belonging to the Association and the regulations under which they may be inspected or borrowed.

The Librarian will gladly receive and acknowledge in the *Gazette* any donation of ancient or modern works on mathematical subjects.

Wanted by purchase or exchange:

1 or 2 copies of	*Gazette*	No. 2 (very important).
1 copy	„	No. 3.
2 or 3 copies of	Annual Report	No. 11 (very important).
1 or 2 „	„	Nos. 10, 12 (very important).
1 copy	„	Nos. 1, 2.

GLASGOW: PRINTED AT THE UNIVERSITY PRESS BY ROBERT MACLEHOSE AND CO. LTD.